Topology in physics 2019, exercises for lecture 10

- The hand-in exercise is exercises 1.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf, readable!)
- Deadline is Wednesday April 24, 23.59.
- Please make sure your name and the week number are present in the file name.
- I may or may not get to the discussion of characteristic classes in the lecture. If I don't, you may want to save exercise 3 until after my next lecture. (Though it can in principle be done already.)

Exercises

$\star$ Exercise 1: Adjoint operators

a. Let $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$ be vector spaces with the usual inner product $\langle a, b\rangle = \overline{a}^T b$. An $m \times n$ matrix A with complex entries can be viewed as an operator from V to W. Show that its adjoint operator $A^\dagger$ is given by the Hermitian conjugate matrix.

b. Let M be an m-dimensional (real) manifold – closed, oriented and without boudary – equipped with a Riemannian metric, and consider the exterior derivative

$$d : \Omega^{r-1}(M) \to \Omega^r(M). \tag{1}$$

Recall the inner product on any $\Omega^s(M)$ is defined as $(\omega, \eta) = \int \omega \wedge \star\eta$. Show that the adjoint operator to d is

$$d^\dagger = (-1)^{mr+m+1} \star d \star . \tag{2}$$

You can use the fact that the Hodge star squares to $(\star)^2 = (-1)^{s(m-s)}$ when acting on $\Omega^s(M)$.

c. Let $D : X \to Y$ be a Fredholm operator, where X and Y are (possibly infinite-dimensional) vector spaces. Show that

$$\text{coker } D \cong \ker D^\dagger. \tag{3}$$

Hint: Decompose $Y = \text{Im}(D) \oplus Z$, where Z is orthogonal to $\text{Im}(D)$. You may assume that $\text{Im}(D)$ is a closed subspace. Show that Z is isomorphic to both coker D and $\ker D^\dagger$.

d. What is the adjoint of the operator $D^\dagger D$? Assuming D maps X to itself, what is the index of the "generalized Laplacian" $\Delta_D = D^\dagger D + DD^\dagger$?

Exercise 2: Why "elliptical"?

Let D be a second order differential operator acting on (real) functions on $\mathbb{R}^2$.

a. Show that the symbol of D can be written in the form

$$\sigma(D;\xi^1,\xi^2) = \alpha(\xi^1)^2 + 2\beta\xi^1\xi^2 + \gamma(\xi^2)^2 \tag{4}$$

b. When does the equation $\sigma(D;\xi^1,\xi^2) = 1$ describe an ellipse, when a hyperbola, and when a straight line in the (ξ^1,ξ^2)-plane? Hint: write this equation in terms of two vectors and a matrix and diagonalize the matrix.

c. Argue that D is an elliptical operator if and only if $\sigma(D;\xi^1,\xi^2) = 1$ describes an ellipse.

Exercise 3: Characteristic classes of dual bundles

Let L be a complex line bundle over an m-dimensional manifold M, and L^* its dual line bundle. (That is: the fibers of L^* are maps from the fibers of L into $\mathbb{C}$.

a. Show that $L \otimes L^*$ is a trivial line bundle. Hint: there are (at least) two ways to do this – you can construct transition functions on $L \otimes L^*$, or you can try to explicitly show that $L \otimes L^*$ has a nonvanishing section and argue that this implies triviality.

b. Use the previous result to argue that $c_1(L^*) = -c_1(L)$.

c. As in the lectures, denote the "diagonal two-forms" one obtains from diagonalizing the curvature two-form $\mathcal{R}$ on $TM^{\mathbb{C}}$ by $x_i(TM^{\mathbb{C}})$, and similarly define $x_i(T^*M^{\mathbb{C}})$. Using the splitting principle, argue that $x_i(T^*M^{\mathbb{C}}) = -x_i(TM^{\mathbb{C}})$. (A detailed proof is not required, a simple few-line argument suffices.)